GROUP 1 PDE PROJECT:
SECTION 5.1 - HEAT ON A BOUNDED DOMAIN

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1. Background - The Motivation and Development of The Heat Equation

Heat, or the thermal energy of a system, is generally defined as the solution of The Heat Equation,

$$u_t = \kappa \Delta u, \ \kappa \text{ is a constant.}$$

It should be noted, however, that this is a special case of a more general PDE which describes the diffusion of some quantity through a medium

$$u_t = \nabla \kappa \nabla u, \ \kappa \text{ is spatially dependent.}$$

For this reason, the Heat Equation is often referred to as The Diffusion Equation. That said, consider the case of a metal sheet made of varying concentrations of different alloys. The Diffusion Equation, as applied to measure heat, cannot be reduced to the general case where κ is constant. From here, the motivation for the solution to a PDE of these types is clear. It is, however, not clear, how such a description for the physical phenomena of heat flow was derived.

To show the derivation of the Heat Equation let us consider how we might describe the following system:

> There is a rod (*see Figure* 1.) of length l, cross sectional area πr^2, and is unifrom in both density ρ and composition(hence, uniform specific heat constant c and thermal conductivity k), with a heat source attached to one end. Assuming that the rod is perfectly laterally insalated, that is to say none of the heat is lost to the surrounding enviroment; and, that the heat source instantainiously heats only one end of the rod. What is the heat of the rod at any given moment and postion after the heat source is turned on and instantly back off?

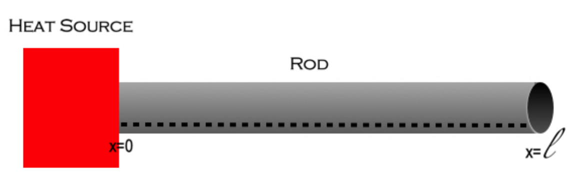

Figure 1. A rod of length l connected to a heat source.

Let's consdier the segment of length Δx, $[x_i,\ x_i{+}\Delta x] = [x_i,\ x_f]$ (*see Figure* 2.), some arbitrary time after the heat source was activated. The total heat within the segment can be written

$$\int_{x_i}^{x_i+\Delta x} c\rho\pi r^2 u(x,t)dx,$$

where $u(x,t)$ is the thermal energy/heat of the rod at some point x and time t. Hence, by the *principle of conservation of heat* we find[1],

$$\begin{aligned} \frac{d}{dt}\int_{x_i}^{x_i+\Delta x} c\rho\pi r^2 u(x,t)dx \ &= \\ c\rho\pi r^2 \int_{x_i}^{x_i+\Delta x} u_t(x,t)dx \ &= \ k\pi r^2[u_x(x_i+\Delta x,t) - u_x(x_i,t)] \ + \\ &\pi r^2 \int_{x_i}^{x_i+\Delta x} f(x,t)dx, \end{aligned} \tag{1}$$

where $f(x,t)$ is some external heat source.

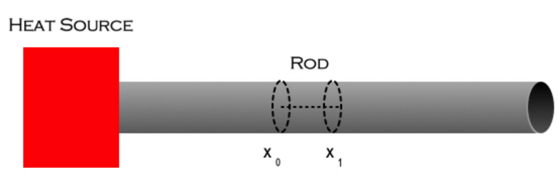

Figure 2.

Replacing πr^2 with A, and applying the Mean Value Theorem[2] we find that equation(1) can be rewritten as,

$$c\rho A u_t(\alpha_1,t) \ = \ kA[u_x(x_i+\Delta x,t) - u_x(x_i,t)] + Af(\alpha_2,t)\Delta x, \tag{2}$$

where $\alpha_1, \alpha_2 \in [x_i, x_f]$. After rearranging our terms and simplifying,

[1]Equation(1) is an application of Fourier's Law

[2]Suppose the function f is continuous on the interval $[a,b]$, then there exists some $\alpha \in [a,b]$ such that $\int_a^b f dx = f(\alpha)(b-a)$

$$u_t(\alpha, t)dx \;=\; \frac{k}{c\rho}[\frac{u_x(x_i + \Delta x, t) - u_x(x_i, t)}{\Delta x}] + \frac{k}{c\rho} f(\alpha, t), \tag{3}$$

Notice, however, we claimed our heat source was instantious. That is, for $t > 0,\;\; f(x, t) = 0$ by design. Hence, in the limit as $\Delta x \to 0$ we see that,

$$u_t(\alpha, t) \;=\; \frac{k}{c\rho}\; u_{xx}(\alpha, t), \tag{4}$$

or,

$$u_t = k u_{xx}$$

Which, as the rod was laterally insalated, we claim that for the given system equation(4) is equivalent to,

$$u_t = \kappa \Delta u.$$

□

2. Example of the Maximum Principle as a Physical Phenomena

Consider the case where a rod is connected to a heat source, as before.

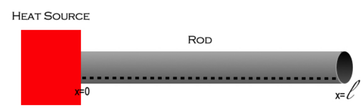

Figure 3. A rod of length l one connected to a heat source

Consider the distribution of heat through the system after some time. We should expect a graph of heat as a function of position at some time t_n, given $u(0,0) = \delta(x)$, to look something like those in the figure 4.

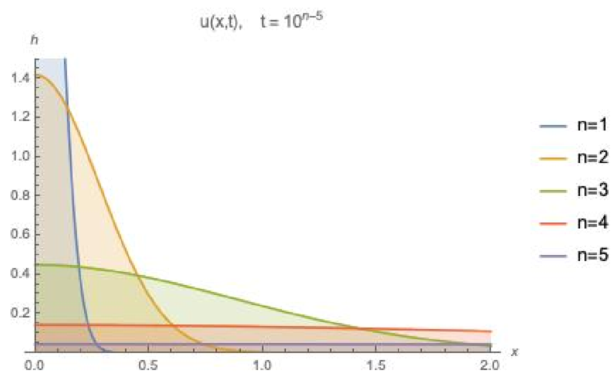

Figure 4. A graph of heat across the length of the rods, at t_n.

Here we can clearly see that for

$$0 < t_n,\ 0 \le x,\ u(0, t_n) \ge u(x, t_n).$$

This can be interpreted as the physical manifestation of the maximum principle.

Similarly, let's consider heat at some point x_n over time. We should expect, a graph of heat as a function of time at some position x_n to look something like those in figure 5.

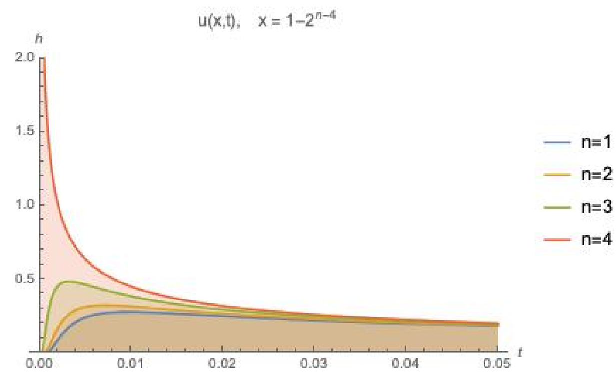

FIGURE 5. A graph of heat at x_n as a function of time

From this we can see that given

$$0 \leq x_n < x_{n+1},\ 0 < t,\ u(0,0) \geq u(x_n,t) \geq u(x_{n+1},t).$$

This too, can be interpreted as the physical manifestation of the maximum principle.

For access to a copy of the notes we collected, as well as a .gif and an .avi video of the evolution of the solution to a 1-D heat equation, you can got to:

https://tuckersideas.net/2019/04/04/heat-equation/