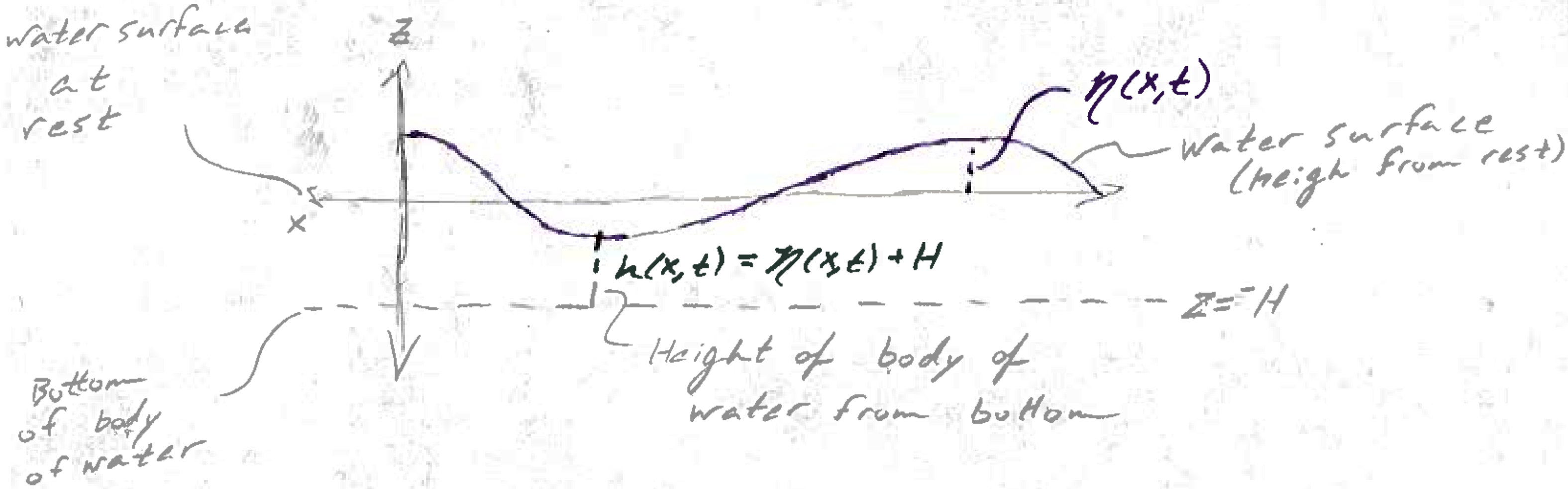


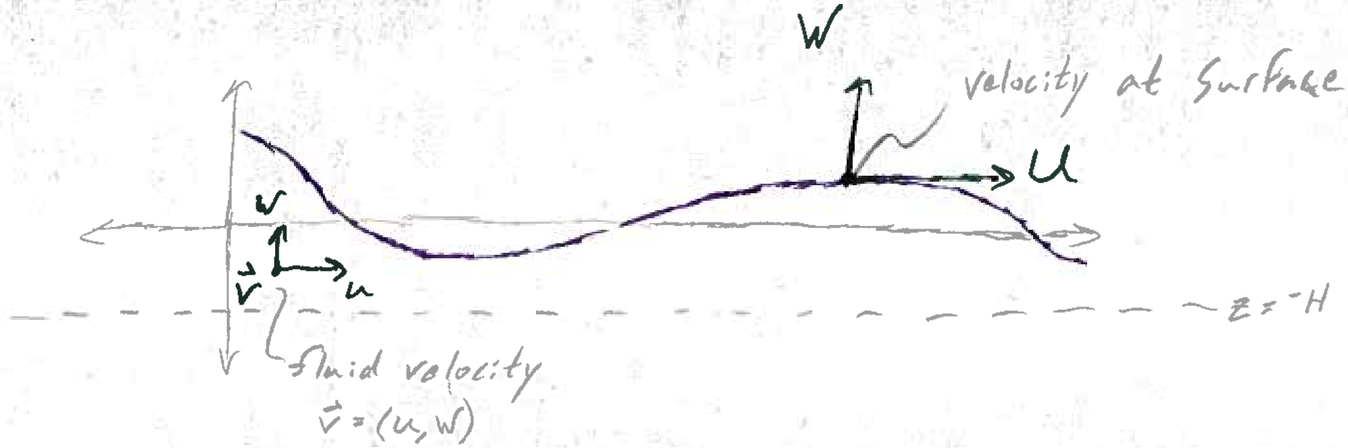
mass of water for $x \in [a,b]$:

$$m_{[a,b]}(t) = \int_a^b \rho\, h(x,t)\, dx = \int_a^b \rho\, (\eta(x,t) + H)\, dx$$

$$= \rho H(b-a) + \rho \int_a^b \eta(x,t)\, dx$$

Equation I) Mass of Water of $x \in [a,b]$

$$m_{[a,b]}(t) = \rho H(b-a) + \rho \int_a^b \eta(x,t)\, dx$$



Water mass flux through line x

$$Q(x,t) = \rho \int_{-H}^{\eta(x,t)} u(x,z,t)\, dz$$

By Conservation of mass

$$D_t m_{[a;b]}(t) = -\left[Q(b,t) - Q(a,t)\right]$$

$$= -\left[\rho \int_{-H}^{\eta(b,t)} u(b,z,t)\, dz - \rho \int_{-H}^{\eta(a,t)} u(a,z,t)\, dz\right]$$

$$= -\rho\left[\int_{-H}^{\eta(b,t)} u(b,z,t)\, dz - \int_{-H}^{\eta(a,t)} u(a,z,t)\, dz\right]$$

Equation 2) Conservation of Mass

$$D_t m_{[a,b]}(t) = -\left[Q(b,t) - Q(a,t)\right]$$

From (1) + (2)

$$\partial_t \int_a^x \rho\, \eta(s,t)\, ds = \frac{d}{dt}\left[m_{[a,x]}(t)\right] = -\left[\int_{-H}^{\eta(x,t)} \rho\, u(x,z,t)\, dz - Q(a,t)\right]$$

So by applying ∂_x to both sides, we find

$$\partial_x \partial_t \int_a^x \rho\, \eta(s,t)\, ds = -\partial_x \int_{-H}^{\eta(x,t)} \rho\, u(x,z,t)\, dz.$$

Rearranging the partials on the left-handside we find

$$\partial_t \partial_x \int_a^x \rho\, \eta(s,t)\, ds = \partial_t \rho\, \eta(x,t).$$

Hence we have arrived at the equation

$$\partial_t \rho\, \eta(x,t) = -\partial_x \int_{-H}^{\eta(x,t)} \rho\, u(x,z,t)\, dz$$

and thus our desired solution

$$\partial_t \eta(x,t) = -\partial_x \int_{-H}^{\eta(x,t)} u(x,z,t)\, dz.$$

Equation 3) Mass Balance

$$\partial_t \eta(x,t) = -\partial_x \int_{-H}^{\eta(x,t)} u(x,z,t)\, dz$$

From the "mass balance" equation

$$\partial_t \eta(x,t) = \partial_x \int_{-H}^{\eta(x,t)} u(x,z,t)\, dz.$$

Expanding the r.h.s. we get

$$\partial_x \int_{-H}^{\eta(x,t)} u(x,z,t)\, dz = u(x,\eta(x,t),t)\,\partial_x \eta(x,t) + \int_{-H}^{\eta(x,t)} \partial_x u(x,z,t)\, dz.$$

By incompressibility

$$\nabla \cdot \vec{v} = \partial_x u + \partial_z w = 0,$$

hence we can write

$$\int_{-H}^{\eta(x,t)} \partial_x u(x,z,t)\, dz = \int_{-H}^{\eta(x,t)} \partial_z w(x,z,t)\, dz.$$

But $w(x,-H,t) = 0$ since the "floor" is impermeable, so

$$\int_{-H}^{\eta(x,t)} \partial_x w(x,z,t)\, dz = w(x,\eta(x,t),t).$$

Thus, using the notation $U(x,t)$ + $W(x,t)$ to denote velocity at the surface, we find

$$\partial_t \eta(x,t) = -U(x,t)\,\partial_x \eta(x,t) + W(x,t) = (U,W)\cdot(-\eta_x, 1).$$

Equation 4)

$$\partial_t \eta(x,t) = -U(x,t)\,\partial_x \eta(x,t) + W(x,t)$$

Observe, the surface can be parameterized by x, at time t, as

$$S(x)\Big|_{@t} = (x, \eta(x,t)).$$

Thus,

$$T(x)\Big|_{@t} = (1, \eta_x(x,t))$$

is the parameterization of its tangent, and

$$N(x)\Big|_{@t} = (-\eta_x(x,t), 1)$$

is the parameterization of its normal.

Hence,

$$\partial_t \eta = (U, W) \cdot (-\eta_x, 1)$$

is describing the evolution of the surface in time as being determined by the velocity normal to the surface.

Interpretation of Eq 4 -

The evolution of the surface is determined by its velocity normal to the surface.

In the case of sufficiently shallow water, the z-dependence in u can be neglected. As a result, the integral on the r.h.s. of the "mass balance" equation yields

$$\int_{-H}^{\eta(x,t)} u(x,t)\,dz = u(x,t)\left(\eta(x,t)+H\right) = u(x,t)h.$$

Hence equation 4 can be simplified to

$$\partial_t \eta(x,t) = -\partial_x\left[h u(x,t)\right].$$

Notice, now, that $\partial_t H = 0$ and so we arrive at the equation

$$\partial_t\left(\eta(x,t)+H\right) = -\partial_x\left[h u(x,t)\right]$$

so

$$\partial_t(h) + \partial_x\left[h u(x,t)\right] = 0, \quad h(x,t) = \eta(x,t) + H.$$

Equation 5) A Continuity Equation + Kinematic Relation for Shallow Water

$$\partial_t\left[h(x,t)\right] + \partial_x\left[h(x,t)u(x,t)\right] = 0.$$

Interpretation)

(5) Seems to be describing the conservation of mass.

This makes sense since we began with C.o.M.

Continuing on with our assumption that for sufficiently shallow water we can neglect the z-dependence of $\vec{v}$, we find that "momentum" can be written

$$M(x,t) = \int_{-H}^{\eta(x,t)} \rho u(x,t)\,dz = \rho u(x,t)\left(\eta(x,t) + H\right) = \rho u(x,t) h.$$

We also find that the "momentum flux" can be written

$$Q(x,t) = \int_{-H}^{\eta(x,t)} \rho u^2(x,t)\,dz = \rho u^2(x,t) h.$$

Note:

"Transport Theorem" states

$$\int_V (\partial_t f + \nabla\cdot\vec{Q} - S)\,dV = 0$$

which yields

$$\partial_t f + \nabla\cdot\vec{Q} = S,$$

where f is the "density" of some property, $\vec{Q}$ is its "flux density," and S is its "source density."

Since pressure along a vertical line can be written

$$P(x,t) = \int_{-H}^{\eta(x,t)} p(x,z,t)\,dz, \quad p(x,z,t) \text{ is the local pressure}$$

by the Transport Theorem we get (P is the source, hence $-\partial_x P$ is the outward source density)

$$\partial_t M(x,t) + \partial_x Q(x,t) = -\partial_x P.$$

Equation 6) Momentum Balance Equation

$$\partial_t\left[\rho h(x,t) u(x,t)\right] + \partial_x\left[\rho h(x,t) u^2(x,t)\right] = -\partial_x \int_{-H}^{\eta(x,t)} p(x,z,t)\,dz.$$

Observe, for shallow water we can approximate $p(x,z,t)$ by the hydrostatic pressure,

$$p(x,z,t) = \rho g(\eta(x,t) - z) + p_{atm}$$

where g is acceleration due to gravity + p_{atm} is atmospheric pressure.

Assuming p_{atm} to be constant in x we can arbitrarily set it to 0.

Evaluating the integral on the R.H.S of (6) we find

$$\int_{-H}^{\eta(x,t)} p(x,z,t)\,dz = \int_{-H}^{\eta} \rho g(\eta - z)\,dz = \rho g\left[\eta z - \tfrac{1}{2}z^2\right]_{-H}^{\eta}$$

$$= \rho g\left[(\eta^2 + H\eta) - \tfrac{1}{2}(\eta^2 - H^2)\right] = \rho g \tfrac{1}{2}\left[\eta^2 + 2\eta H + H^2\right]$$

$$= \tfrac{1}{2}\rho g(\eta + H)^2 = \tfrac{1}{2}\rho g h^2.$$

Thus (6) can be written as

$$\partial_t[\rho h(x,t)u(x,t)] + \partial_x[\rho h(x,t)u^2(x,t)] = -\partial_x\left[\tfrac{1}{2}\rho g h^2(x,t)\right]$$

and by a simple rearrangement we find

$$\partial_t[\rho h(x,t)u(x,t)] = -\partial_x\left[\rho h(x,t)u^2(x,t) + \tfrac{1}{2}\rho g h^2(x,t)\right].$$

Simplifying the equation yields

$$\partial_t[hu] = -\partial_x[hu^2 + \tfrac{1}{2}gh^2].$$

Equation 7)

$$\partial_t[hu] = -\partial_x[hu^2 + \tfrac{1}{2}gh^2]$$

Expanding both sides of (7) we get

$$\partial_t(hu) = -\partial_x\left(hu^2 + \tfrac{1}{2}gh^2\right)$$

$$u\partial_t(h) + h\partial_t(u) = -\left(u^2\partial_x h + 2uh\partial_x u + gh\partial_x h\right).$$

and by application of (5) we find

$$u(-\partial_x[hu]) + h\partial_t(u) = -\left(u^2\partial_x h + 2uh\partial_x u + gh\partial_x h\right)$$

$$u^-(u\partial_x h + h\partial_x u) + h\partial_t u = -\left(u^2\partial_x h + 2uh\partial_x u + gh\partial_x h\right)$$

$$-\left(u^2\partial_x h + uh\partial_x u\right) + h\partial_t u = -\left(u^2\partial_x h + 2uh\partial_x u + gh\partial_x h\right).$$

By rearranging our terms we get

$$h\partial_t u = -\left(u^2\partial_x h + 2uh\partial_x u + gh\partial_x h\right) + \left(u^2\partial_x h + uh\partial_x u\right)$$

$$h\partial_t u = -\left(uh\partial_x u + gh\partial_x h\right).$$

So, finally, by simplification we get that

$$\partial_t u = -\left(u\partial_x u + g\partial_x h\right)$$

or

$$\partial_t u = -\partial_x\left(\tfrac{1}{2}u^2 + gh\right).$$

Equation 8)

$$\partial_t u = -\partial_x\left(\tfrac{1}{2}u^2 + gh\right)$$

Observe, we can interpret (8) as

* the Lagrangian changes in x opposite of velocity in time

\- or equivalently -

* as a conservation law for u where $\frac{1}{2}u^2+gh$ is its flux

Also observe,

$$\begin{cases} \partial_t h = \partial_x (hu) \\ \partial_t u = -\partial_x \left(\frac{1}{2}u^2 + gh\right) \end{cases}$$

form a closed system of equations which describe the evolution of shallow water waves.